Red Level

1) (**5 credits**) Sheldon Cooper reads his 10 digit mobile phone number aloud to his friends. Howard Wolowitz and Raj Koothrappali failed to write down some digits: Howard got '8248312', and Raj got '38128432'. Restore Sheldon's phone number [taking into account that each digit is repeated in his number at most two times]. (Please find all alternative solutions) PLEASE CONSIDER TO WHAT EXTENT THE CONDITION IN SQUARE BRACKETS IS USEFUL TO SIMPLIFY THE EXPLANATIONS (it does not affect the answer)

Solution: Both records of the phone number written down by Sheldon's friends contain two eights each, and this mean that the original number contains two eights. The first eight should be followed at least by 3, the digits 1, 2 and 4 should be between two eights, and the second eight should be followed by the digits 1, 2, 3 and 4 (because they are in the corresponding intervals at least in one record). Now we have 2 eights and at least 8 other digits; since the number consists of 10 digits according to the statement, there are no other digits in it. 1 should be before 2 between the eights (see Raj's record), and 2 should be before 4 between them (see Howard's record), 4 should be before 3 after the eights (see Howard's record), and 1 and 2 should be after 3 (see Raj's record). Therefore, the **answer** is 3812484312.

2) (**6 credits**) Forty thieves filled one of their chests with gold and silver dust to the top; the quantity of gold dust was 2 times more than the quantity of silver dust. According to calculations of Ali Baba, after pouring one-half of the quantity of silver dust out of the chest and filling the chest with gold dust to the top, the cost of dust in this chest will increase by 20 percent. What is the percentage decrease if Ali Baba pours one-half of the quantity of gold dust out of the chest and fills the chest with silver dust to the top?

Solution: Let the cost of gold dust filling the sixth part of the chest is x, and the cost of silver dust filling the same volume is y. This means that in the first case the cost of dust in the chest increases by $x - y$, and in the second case (when we replace gold with silver in two sixths, i.e., in one third of the chest) it decreases by $2(x - y)$, i.e. the cost decrease in the second case is twice as much than the cost increase in the first case. Therefore, since the cost increased by 20 percent in the first case, then in the second case it decreased by 40 percent.

3) (**8 credits**) Each of brothers Grimm buys a pastry or a tub of ice cream every day. The younger brother always buys the food bought by his elder brother a week ago, and the elder brother never buys the food bought by his younger brother a week ago. What is the maximum number of tubs of ice cream that could be bought by the elder brother in November?

Solution: Note that the food bought by the elder brother on the day X differs from the food bought by him on the day X+14, because the food bought by him on the day X is the same as the food bought by the younger brother on the day X+7, and the food bought by him on the day X+14 is not the same. Therefore, on the days of each pair (1.11, 15.11) (2.11, 16.11), (3.11, 17.11),... (14.11, 28.11) the elder brother ate one tub of ice cream, i.e. he ate 14 tubs from November 1 to November 28. He could also eat another two tubs of ice cream on November 29 or 30, i.e. the maximum number of tubs is 16. If the elder brother eats according to the schedule 'two ice cream weeks, two pastry weeks' and the next ice cream week begins on November 1 and the younger brother eats in accordance with his schedule with one week shift in dates, the answer mentioned in the previous sentence is true, i.e. the option 16 is possible and it is the solution of this problem.

4) (**8 credits**) In the triangle ABC, the value of the angle C is three times greater than the value of the angle A, and the side AB is twice as long as the side BC. Find the value of the angle B.

Solution: after we draw the median M, we get the isosceles triangle *CBM*. The value of the angle B is $180° - \angle A - \angle C = 180° - 4\angle A$, therefore the value of each of two equal angles at the base CM is $2\angle A$. This means that $\angle MCA = \angle ACB - \angle BCM = 3\angle A - 2\angle A = \angle A$. Therefore, the triangle *ACM* is isosceles and $CM = MA = MB$, i.e. the triangle *BCM* is equilateral and $\angle B = 60°$.

5) (**10 credits**) 150 employees (in total) worked in a two-storey office. When each male office employee sent a message to a female employee working on another floor, thirty female office employees did not receive any message, and each of other female employees received one message. At the same time, the number of those first-floor female employees, who received the message, is less by half than the total number of employees on the first floor, and the number of those second-floor female employees, who received the message, is equal to 2/7 of the total number of employees on the second floor. How many male employees worked on the second floor?

Solution: Let $2x$ male employees work on the first floor and y male employees work on the second floor. This means that y female employees working on the first floor received the message and other y office employees (including $y — 2x$ female employees) did not receive it. Similarly, $2x$ female employees working on the second floor received the message and other $5x$ office employees (including $5x — y$ female employees) did not receive it. The total number of female employees who did not receive the message was $y — 2x + 5x — y = 3x$. According to the statement, 30 female office employees did not receive the message, from which $x = 10$ and the number of male employees working on the first floor is $2x = 20$.

The total number of those who sent the message and those who received it was 150 — 30 = 120; half of them (i.e. 60) were male employees, therefore, 60 — 20 = 40 male employees worked on the second floor.

6) (**10 credits**) $x + y + z = 0$. Suppose you solved that equation and found x, y and z. Now consider $sin(x)$. What is the largest number of positive numbers among

$$sin(x), sin(y), sin(z), cos(x), cos(y), cos(z)?$$

Answer: 5.

Solution: Here is one of the samples where the number of positive numbers is the largest: $x = y = \pi/6$, $z = — \pi/3$; in this case $sin(z)$ is negative, and all other mentioned numbers are positive. All six numbers cannot be positive (in particular, because $sin(z) = —sin(—z) = —(sin(x)cos(y) + sin(y)cos(x))$; therefore, if $sin(x)$, $cos(y)$, $sin(y)$, $cos(x)$ are positive, then $sin(z)$ is surely negative). This problem may also be solved not through the use the "sine of the sum" formula but through the analysis of the intervals where the sine and the cosine are positive, though this solution is a little longer.

7) (**12 credits**) Artemis Fowl selected four natural numbers. Then he calculated the product of the first, second and fourth numbers, the product of the first, third and fourth numbers, the product of the second, third and fourth numbers and finally the sum of the first, second and third numbers. After that, he wrote the results on the board in the ascending order: 24, 27, 120, 160. Restore the numbers selected by Artemis in the correct order.

Solution: Let us denote the initial numbers by a, b, c, d. Note that the product of any two of three products abd, acd, bed is divisible by the third product. On the other hand, neither product of any two of three numbers 24, 120, 160 is divisible by 27 (when resolving the latter into prime factors we obtain 3 threes, and when resolving the first three numbers we obtain not more than 1 prime factor in each case). Therefore, 27 is a sum of first three numbers, and the ratio of two of them is equal to the ratio of the products 24/160 = 3/20. Since both these numbers are less than 27, this can only happen if one of them is 3 and the other is 20. This means that we may get one more number: 27 — 20 — 3 = 4, and the fourth number can be obtained, for example, in such way: $160/(20\times4) = 2$. There is also a more awkward solution: we may analyze all 4 options, find proportions for first three numbers for each case and then sum up.

8) (**12 credits**) Ulysses moved from the point A to the point B along the polygonal chain ACB consisting of 2 segments. When moving along it, Ulysses constantly moves away from the point A and becomes closer to the point B. Any longer polygonal chain of 2 segments connecting the points A and B does not meet such condition. Find the value of the angle ACB.

Answer: 90 degrees.

Solution: Let us draw a perpendicular to the line AC through D; D is its foot. It is clear that when moving from A in the direction away from D or when moving from D in the direction away from A, we move away from B. This means that C is on the segment AD. But if C does not coincide with D, the polygonal chain ADB is longer, but it apparently meets the same requirements. Therefore, $C = D$ and the angle ACB is right.

9) (**14 credits**) Scrooge McDuck has 19 coins: 18 identical real coins and 1 fake coin (its weight is a little less than the weight of a real coin). He also has 2 two-pan scales; each of them can show the bowl on which coins are heavier, but in this case it breaks due to imbalance. Scrooge has developed an algorithm which makes it possible to find a fake coin in three weighings, even if such weighings result in the breakage of the scales. How many coins should he put on the bowls (such action is the first stage of his algorithm)? [Apparently, the uniqueness of this solution does not need to be proved.]

Solution: First of all, Scrooge McDuck has to put five coins on each bowl. If the scale is in balance, he has to find a fake coin among 9 remaining coins. It can be found as usually in two weighings (to divide all coins into three groups, to place two of those groups on each of two pans, and then to weigh two of three doubtful coins). Since he still has two scales operable, he is able to make two weighings in any case.

If the scale is not in balance, he gets a doubtful group of five coins and one scale remains operable. Now he has to put one coin selected from this group on each pan. He either finds the fake coin (and breaks the last scale), or the scale remains operable giving him the possibility to weigh two of three remaining doubtful coins and to identify the fake coin unambiguously.

10) (**15 credits**) Each cell of the 100 x 100 board contains a natural number, and all such numbers are different. The White King wants to pass through several squares of the board (he chooses the starting point at his discretion) provided that the number contained in each next square is greater than the number contained in the previous one. What is the maximum number of squares that can be visited by the King for sure (regardless of the arrangement of the numbers)?

Answer: 4.

Solution: Note that a chess king can pass through squares of any 2x2 part in any order (because it can move from any square of this part to any other square), in particular, through squares with ascending numbers.

Let us analyze the example showing that the King can pass through four or less squares only. We divide the board into 2x2 squares, and then write the numbers from 1 to 2500 in top-left corners of those 2x2 parts (let them be denoted as Type 1 Squares), the numbers from 2501 to 5000 in their top-right corners (Type 2 Squares), the numbers from 5001 to 7500 in their bottom-left corners (Type 3 Squares), and the numbers from 7501 to 10000 in their bottom-right corners (Type 4 Squares). Note that the King cannot pass from any cell to the cell of the same type, and this means that if he wants to pass through ascending numbers, he has to move to the square of a greater type. His path can apparently include three moves only.